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AUTHOR(S):

Aihaiti, Abulizi

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Large time behavior of solutions to the compressible Navier-Stokes equations in an infinite layer under slip boundary condition *

Abulizi Aihaiti

Graduate School of Mathematics, Kyushu University

1 Introduction

We study the large time behavior of solutions of the compressible Navier-Stokes equations

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad (1)$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu') \nabla \operatorname{div} v + \nabla P(\rho) = 0 \quad (2)$$

in an infinite layer Ω of \mathbb{R}^2 :

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2; x_1 \in \mathbb{R}, 0 < x_2 < 1\}$$

under the slip boundary condition

$$\partial_{x_2} v^1|_{x_2=0,1} = 0, \quad v^2|_{x_2=0,1} = 0. \quad (3)$$

Here $\rho = \rho(x, t) > 0$ and $v = {}^T(v^1(x, t), v^2(x, t))$ denote the unknown density and velocity, respectively, at time $t \geq 0$ and position $x \in \Omega$; $P = P(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying $P'(\rho_*) > 0$ for a given constant $\rho_* > 0$; μ and μ' are viscosity coefficients that are assumed to be constants and satisfy $\mu > 0, \mu + \mu' \geq 0$; div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x . Here and in what follows ${}^T \cdot$ means the transposition.

We impose the initial condition

$$\rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0. \quad (4)$$

Here $\rho_0 = \rho_0(x)$ and $v_0 = v_0(x)$ satisfy $\rho_0(x) \rightarrow \rho_*$ and $v_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

The aim of our research is to investigate the large time behavior of solutions to (1)-(4) around the motionless state $\rho = \rho_*$, $v = 0$. We rewrite (1)-(2) into the following equations for the perturbation

$$\partial_t \phi + \gamma \operatorname{div} w = f^0(\phi, w), \quad (5)$$

*This is based on a joint work with Shota Enomoto (Graduate School of Mathematics, Kyushu University) and Professor Yoshiyuki Kagei (Faculty of Mathematics, Kyushu University).

$$\partial_t w - \nu \Delta w - \tilde{\nu} \nabla \operatorname{div} w + \gamma \nabla \phi = \tilde{f}(\phi, w). \quad (6)$$

Here $u = {}^\top(\phi, w)$ with $\phi = \frac{1}{\rho_*}(\rho - \rho_*)$ and $w = \frac{1}{\gamma}v$ denotes the perturbation from $u_s = {}^\top(\rho_*, 0)$; ν , $\tilde{\nu}$ and γ are parameters given by

$$\nu = \frac{\mu}{\rho_*}, \quad \tilde{\nu} = \frac{\mu + \mu'}{\rho_*}, \quad \gamma = \sqrt{P'(\rho_*)};$$

and $f(\phi, w) = {}^\top(f^0(\phi, w), \tilde{f}(\phi, w))$ denote the nonlinear terms.

The boundary condition (3) and initial condition (4) are transformed into

$$\partial_{x_2} w^1|_{x_2=0,1} = 0, \quad w^2|_{x_2=0,1} = 0 \quad (7)$$

and

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \quad (8)$$

Here u_0 satisfies $u_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

The large time behavior of solutions of the compressible Navier-Stokes equations (1)-(2) on the layer Ω was studied in [1, 2, 3, 4] under the non-slip boundary condition $v|_{x_2=0,1} = 0$. It was shown in [3] that the large time behavior of perturbations of the motionless state is described by a one-dimensional linear heat equation. In [4] the asymptotic stability of parallel flow was considered and it was proved that the large time behavior of perturbations of parallel flow is described by a one-dimensional viscous Burgers equation when the Reynolds and Mach numbers are sufficiently small. In the case of time-periodic parallel flow, the large time behavior of perturbations is also described by a one-dimensional diffusion equation ([1, 2]). In all cases of [1, 2, 3, 4], the asymptotic leading parts under the non-slip boundary condition exhibit purely diffusive phenomena. In this paper we show that the solution of (1)-(2) under the slip boundary condition (3) with (4) behaves like a superposition of one-dimensional diffusion waves as $t \rightarrow \infty$ as in the case of one-dimensional compressible Navier-Stokes equation [7, 10]. More precisely, consider the problem (5)-(8) for u . We prove that, under appropriate conditions for u_0 , the solution $u(t)$ satisfies

$$\|\partial_x^k(u - \chi_+ \mathbf{a}_+ - \chi_- \mathbf{a}_-)(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1, \quad (9)$$

where $\mathbf{a}_\pm = {}^\top(1, \pm 1, 0)$ and $\chi_\pm = \chi_\pm(x_1, t)$ are the diffusion waves given by

$$\chi_\pm(x_1, t) = z_\pm(x_1 \pm \gamma t, t). \quad (10)$$

Here $z_\pm = z_\pm(x_1, t)$ are the self-similar solutions of the viscous Burgers equations

$$\partial_t z_\pm - \frac{\nu + \tilde{\nu}}{2} \partial_{x_1}^2 z_\pm \mp c \partial_{x_1}(z_\pm^2) = 0 \quad (11)$$

satisfying

$$\int_{\mathbb{R}} z_\pm(x_1, t) dx_1 = \frac{1}{2} \int_{\Omega} (\phi_0(x) \pm (1 + \phi_0(x)) w_0^1(x)) dx \quad (12)$$

for some constant $c \in \mathbb{R}$.

In contrast to the case of the non-slip boundary condition, we see that a hyperbolic aspect of (1)-(2) appears in the asymptotic leading part of the solution under the slip boundary condition.

2 Main results

We set

$$H_*^2 = \{w = {}^\top(w^1, w^2) \in H^2(\Omega); \partial_{x_2} w^1|_{x_2=0,1} = 0, w^2|_{x_2=0,1} = 0\}.$$

For $\alpha \in \mathbb{R}$, we denote by $L_\alpha^1 = L_\alpha^1(\Omega)$ the weighted L^1 space with weight $(1 + |x_1|)^\alpha$, and its norm is denoted by

$$\|f\|_{L_\alpha^1} = \int_\Omega (1 + |x_1|)^\alpha |f(x)| dx.$$

We now state the main results of this paper. We have the following decay estimate of the L^2 norm of the solution u .

Theorem 2.1 *There exists a positive number ε_0 such that if $u_0 = {}^\top(\phi_0, w_0) \in (H^2 \times H_*^2) \cap L^1$ with $w_0 = {}^\top(w_0^1, w_0^2)$ satisfies $\|u_0\|_{H^2 \cap L^1} \leq \varepsilon_0$, then problem (5)-(8) has a unique global solution*

$$u(t) = {}^\top(\phi(t), w(t)) \in C([0, \infty); H^2 \times H_*^2)$$

and $u(t)$ satisfies

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1}$$

for $t \geq 0$, $k = 0, 1, 2$.

We next consider the asymptotic behavior of solutions.

Theorem 2.2 *In addition to the assumptions of Theorem 2.1, if $\phi_0, w_0^1 \in L_{1/2}^1$, then*

$$\|\partial_x^k (u - \chi_+ \mathbf{a}_+ - \chi_- \mathbf{a}_-)(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1.$$

Here $\mathbf{a}_\pm = {}^\top(1, \pm 1, 0)$ and $\chi_\pm = \chi_\pm(x_1, t)$ are the diffusion waves given in (10)-(12).

3 Outline of the proof

3.1 Spectral properties of linearized operator

We rewrite the equation (5)-(6) as following

$$\partial_t u + Lu = F, \quad u|_{t=0} = u_0, \tag{13}$$

where $u = {}^\top(\phi, w)$; $F = {}^\top(f^0, \tilde{f})$ with $\tilde{f} = {}^\top(f^1, f^2)$ is a given function, and L is an operator of the form

$$L = \begin{pmatrix} 0 & \gamma \operatorname{div} \\ \gamma \nabla & -\nu \Delta - \tilde{\nu} \nabla \operatorname{div} \end{pmatrix}$$

in $H^1 \times L^2$ with domain $D(L) = H^1 \times H_*^2$.

To investigate (13), we take the Fourier transform of (13) in $x_1 \in \mathbb{R}$, and then we expand \hat{u} and \hat{F} into the Fourier series to obtain

$$\partial_t \hat{u}_k + \hat{L}_{\xi,k} \hat{u}_k = \hat{F}_k, \quad (14)$$

where $\hat{u}_k = {}^\top(\hat{\phi}_k, \hat{w}_k^1, \hat{w}_k^2)$, $\hat{F}_k = {}^\top(\hat{f}_k^0, \hat{f}_k^1, \hat{f}_k^2)$ and

$$\hat{L}_{\xi,k} = \begin{pmatrix} 0 & i\gamma\xi & \gamma k\pi \\ i\gamma\xi & \nu(\xi^2 + k^2\pi^2) + \tilde{\nu}\xi^2 & -i\tilde{\nu}k\pi\xi \\ -\gamma k\pi & i\tilde{\nu}k\pi\xi & \nu(\xi^2 + k^2\pi^2) + \tilde{\nu}k^2\pi^2 \end{pmatrix}.$$

For the the spectrum of $\sigma(-\hat{L}_{\xi,k})$, the case $|\xi| \ll 1$, $k = 0$ is the slowest decay part. In this case, the eigenvalues and eigenprojections are given by

$$\begin{aligned} \lambda_{\pm,0}(\xi) &= \pm i\gamma\xi - \frac{\nu + \tilde{\nu}}{2}\xi^2 + O(\xi^3) \quad (\xi \rightarrow 0), \\ P_{\pm,\xi} &= \tilde{P}_{\pm}(1 + O(\xi))\Pi, \end{aligned}$$

where

$$\tilde{P}_{\pm} = \begin{pmatrix} 1 & \pm 1 & 0 \\ \pm 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi u = \begin{pmatrix} \langle \phi \rangle \\ \langle w^1 \rangle \\ 0 \end{pmatrix}.$$

Here $u = {}^\top(\phi, w^1, w^2)$ and $\langle \phi \rangle$ is defined by $\langle \phi \rangle = \int_0^1 \phi(x_2) dx_2$.

3.2 Decay estimate: Proof of Theorem 2.1

We consider the nonlinear problem

$$\begin{cases} \partial_t u + Lu = F(u), \\ u|_{t=0} = u_0. \end{cases} \quad (15)$$

Here $u = {}^\top(\phi, w)$ and $F(u) = {}^\top(f^0(\phi, w), \tilde{f}(\phi, w))$.

One can prove the local solvability for (15) as in [5].

Proposition 3.1 *Assume that $u_0 = {}^\top(\phi_0, w_0) \in H^2 \times H_*^2$ and $\|\phi_0\|_\infty \leq \frac{1}{2}$. Then there exists $T_0 > 0$ depending on $\|u_0\|_{H^2}$ such that problem (15) has a unique solution $u = {}^\top(\phi, w)$ on $[0, T_0]$ satisfying $u \in C([0, T_0]; H^2 \times H_*^2) \cap C^1([0, T_0]; L^2)$ with $w \in L^2(0, T_0; H^3)$ and $\|\phi_0(t)\|_\infty \leq \frac{3}{4}$ for $t \in [0, T_0]$. Furthermore, the inequality*

$$\sup_{t \in [0, T_0]} \{\|u(t)\|_{H^2} + \|\partial_t u(t)\|_{L^2}\} + \int_0^{T_0} \|w\|_{H^3}^2 dt \leq C_0 \{1 + \|u_0\|_{H^2}^2\}^a \|u_0\|_{H^2}^2 \quad (16)$$

holds with some constants $C_0 > 0$ and $a > 0$.

The global existence of $u(t)$ follows in a standard manner from Proposition 3.1 and Proposition 3.4 below which provides the a priori bound $\|u(t)\|_{H^2} \leq C\|u_0\|_{H^2 \cap L^1}$ when $\|u_0\|_{H^2 \cap L^1}$ is sufficiently small.

We next consider the a priori estimates for $u(t)$. Let r_0 be a number satisfying $0 < r_0 \leq 1$. We introduce the cut-off function $\mathbf{1}_{\{|\xi| \leq r_0\}}$ defined by

$$\mathbf{1}_{\{|\xi| \leq r_0\}} = \begin{cases} 1 & (|\xi| < r_0), \\ 0 & (|\xi| \geq r_0). \end{cases}$$

We introduce the projections P_1 and P_∞ defined by

$$P_1 u = \mathcal{F}^{-1} \mathbf{1}_{\{|\xi| \leq r_0\}} \Pi \mathcal{F} u, \quad P_\infty = I - P_1.$$

We decompose $u = {}^\top(\phi, w)$ into

$$u = u_1 + u_\infty,$$

where

$$u_1 = P_1 u = {}^\top(\phi_1, w_1^1, w_1^2), \quad u_\infty = P_\infty u = {}^\top(\phi_\infty, w_\infty^1, w_\infty^2).$$

Proposition 3.2 *Let $u(t)$ be a solution of (15) on $[0, T]$. Assume that $u \in C([0, T]; H^2 \times H_*^2) \cap C^1([0, T]; L^2)$ with $w \in L^2(0, T; H^3)$. Then*

$$u_1 = {}^\top(\phi_1, w_1) \in C^1([0, T]; H^l(\Omega)) \quad (\forall l = 0, 1, 2, \dots)$$

and

$$u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0, T]; H^2 \times H_*^2) \cap C^1([0, T]; L^2)$$

with $w_\infty \in L^2(0, T; H^3)$.

Furthermore, u_1 and u_∞ satisfy

$$u_1 = P_1 e^{-tL} u_0 + \int_0^t P_1 e^{-(t-\tau)L} F(u(\tau)) d\tau, \quad (17)$$

$$\partial_t u_\infty + L u_\infty = F_\infty, \quad u_\infty|_{t=0} = P_\infty u_0, \quad (18)$$

where $F_\infty = P_\infty F = {}^\top(f_\infty^0, \tilde{f}_\infty)$, $\tilde{f}_\infty = (f_\infty^1, f_\infty^2)$.

We define $M(t) \geq 0$ by

$$M(t) = M_1(t) + M_\infty(t) \quad (t \in [0, T]).$$

Here $M_1(t)$ and $M_\infty(t)$ are defined by

$$M_1(t) = \sup_{0 \leq \tau \leq t} \left\{ \sum_{k=0}^2 (1+\tau)^{\frac{1}{4} + \frac{k}{2}} \|\partial_{x_1}^k u_1(\tau)\|_{L^2} + (1+\tau)^{\frac{3}{4}} \|\partial_t u_1(\tau)\|_{L^2} \right\},$$

$$M_\infty(t) = \left(\sup_{0 \leq \tau \leq t} (1+\tau)^{\frac{5}{2}} \{ \|u_\infty(\tau)\|_{H^2}^2 + \|\partial_t u_\infty(\tau)\|_{L^2}^2 \} \right)^{\frac{1}{2}}.$$

We introduce the quantities $E_\infty(t)$ and $D_\infty(t)$ for $u_\infty(t) = {}^\top(\phi_\infty(t), w_\infty(t))$:

$$\begin{aligned} E_\infty(t) &= \|u_\infty(t)\|_{H^2}^2 + \|\partial_t u_\infty(t)\|_{L^2}^2, \\ D_\infty(t) &= \|\nabla \phi_\infty(t)\|_{H^1}^2 + \|\nabla w_\infty(t)\|_{H^2}^2 + \|\partial_t u_\infty(t)\|_{H^1}^2. \end{aligned}$$

Proposition 3.3 *Let $u(t)$ be a solution of (15) on $[0, T]$. Then there exists a positive constant ε_1 such that if $\|u(t)\|_{H^2} \leq \varepsilon_1$ and $M(t) \leq 1$ for $t \in [0, T]$, the estimates*

$$M_1(t) \leq C\{\|u_0\|_{L^1} + M(t)^2\} \quad (19)$$

and

$$\begin{aligned} E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau) d\tau \\ \leq C\{e^{-at} E_\infty(0) + (1+t)^{-\frac{5}{2}} M(t)^4 + \int_0^t e^{-a(t-\tau)} \mathcal{R}(\tau) d\tau\} \end{aligned} \quad (20)$$

hold uniformly for $t \in [0, T]$ with $C > 0$ independent of T . Here $a = a(\nu, \bar{\nu}, \gamma)$ is a positive constant; and $\mathcal{R}(t)$ is a function satisfying the estimate

$$\mathcal{R}(t) \leq C\{(1+t)^{-\frac{5}{2}} M(t)^3 + M(t) D_\infty(t)\}. \quad (21)$$

3.3 Estimates of low and high frequency parts

We see from spectral properties of $-\hat{L}_{\xi,k}$ and the definition of Π that

$$\|\partial_{x_1}^l e^{-tL} P_1 u_0\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_{L^1} \quad (22)$$

for $l \geq 0$, and we thus obtain

$$\|\partial_{x_1}^k u_1(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \{\|u_0\|_{L^1} + M(t)^2\} \quad (23)$$

for $k = 0, 1, 2$.

As for the time derivative, we have

$$\|\partial_t u_1(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}} \{\|u_0\|_{L^1} + M(t)^2\}. \quad (24)$$

By (23) and (24), we obtain (19).

As for the high-frequency part $u_\infty = P_\infty u$, we apply the Matsumura-Nishida energy method to prove estimate (20) in Proposition 3.3.

From Proposition 3.3, one can show the following uniform estimate of $M(t)$ as in [4].

Proposition 3.4 *If $\|u_0\|_{H^2 \cap L^1}$ is sufficiently small, then*

$$M(t) \leq C\|u_0\|_{H^2 \cap L^1}. \quad (25)$$

Theorem 2.1 now follows from Propositions 3.1 and 3.4.

3.4 Asymptotic behavior: Proof of Theorem 2.2

To prove Theorem 2.2 we rewrite (1)-(2) in the form of conservation laws.

We set

$$m = \rho v = \rho_*(1 + \phi)v.$$

Then (1)-(2) is written as

$$\begin{cases} \partial_t \rho + \operatorname{div} m = 0, \\ \partial_t m - \mu \Delta(\frac{m}{\rho}) - (\mu + \mu') \nabla \operatorname{div}(\frac{m}{\rho}) + \nabla P(\rho) + \operatorname{div}(\frac{m \otimes m}{\rho}) = 0, \end{cases} \quad (26)$$

and the boundary condition (3) is transformed into

$$\partial_{x_2} \left(\frac{m^1}{\rho} \right) \Big|_{x_2=0,1} = 0, \quad m^2 \Big|_{x_2=0,1} = 0. \quad (27)$$

We decompose ${}^\top(\phi, m^1)$ as

$$\begin{aligned} \phi &= \Phi + \Phi_\infty, \quad \Phi = \phi_1 = \tilde{P}_1 \phi, \quad \Phi_\infty = \phi_\infty = \tilde{P}_\infty \phi, \\ m^1 &= \rho_* \gamma (M + M_\infty), \quad M = \frac{1}{\rho_* \gamma} \tilde{P}_1 m^1, \quad M_\infty = \frac{1}{\rho_* \gamma} \tilde{P}_\infty m^1. \end{aligned}$$

Note that $w^1 = \frac{M+M_\infty}{1+\phi}$. Here the operators \tilde{P}_1 and \tilde{P}_∞ defined by

$$\tilde{P}_1 \phi = \mathcal{F}^{-1} \mathbf{1}_{\{|\xi| \leq r_0\}} \langle \mathcal{F} \phi \rangle, \quad \tilde{P}_\infty = I - \tilde{P}_1.$$

Applying P_1 to (26) and using (27), we have

$$\begin{cases} \partial_t \Phi + \gamma \partial_{x_1} M = 0, \\ \partial_t M - (\nu + \tilde{\nu}) \partial_{x_1}^2 M + \gamma \partial_{x_1} \Phi = \partial_{x_1} \tilde{P}_1 g(U) + \partial_{x_1} \tilde{P}_1 \tilde{g}. \end{cases} \quad (28)$$

Here $U = {}^\top(\Phi, M)$,

$$\begin{aligned} g(U) &= -\frac{\rho_* P''(\rho_*)}{2\gamma} \Phi^2 - \gamma M^2, \\ \tilde{g} &= \tilde{g}(x, t) = -(\nu + \tilde{\nu}) \partial_{x_1} (\phi w^1) - \frac{\rho_* P''(\rho_*)}{2\gamma} (2\Phi \Phi_\infty + \Phi_\infty^2) \\ &\quad - \gamma (2M M_\infty + M_\infty^2) + \gamma (\phi w^1 (M + M_\infty)), \end{aligned}$$

where $\phi = \Phi + \Phi_\infty$, $w^1 = \frac{M+M_\infty}{1+\phi}$.

We write (28) in the form

$$\begin{cases} \partial_t U + L_0 U = \partial_{x_1} P_0 G(U) + \partial_{x_1} P_0 \tilde{G}, \quad U = P_0 U, \\ U|_{t=0} = P_0 U_0, \end{cases} \quad (29)$$

where $U_0 = {}^\top(\phi_0, \frac{1}{\rho_* \gamma} m_0^1) = {}^\top(\phi_0, (1 + \phi_0) w_0^1)$,

$$L_0 = \begin{pmatrix} 0 & \gamma \partial_{x_1} \\ \gamma \partial_{x_1} & -(\nu + \tilde{\nu}) \partial_{x_1}^2 \end{pmatrix},$$

$$G(U) = \begin{pmatrix} 0 \\ g(U) \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix},$$

and P_0 denotes the projection defined by

$$P_0(U) = \begin{pmatrix} \tilde{P}_1 \Phi \\ \tilde{P}_1 M \end{pmatrix}$$

for $U = {}^\top(\Phi, M)$.

We see from the spectral properties of $-\hat{L}_{\xi,k}$ that

$$e^{-tL_0} = \mathcal{F}^{-1}(e^{\lambda_+ t} P_+ + e^{\lambda_- t} P_-) \mathcal{F},$$

where

$$\lambda_{\pm} = \lambda_{\pm,0} = -\frac{1}{2}(\nu + \tilde{\nu})\xi^2 \pm \frac{1}{2}\sqrt{(\nu + \tilde{\nu})^2\xi^4 - 4\gamma^2\xi^2},$$

$$P_{\pm} = \pm \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} -\lambda_{\mp} & i\gamma\xi \\ i\gamma\xi & \lambda_{\pm} \end{pmatrix}.$$

We observe that, for $|\xi| \ll 1$,

$$\lambda_{\pm} = -\frac{\nu + \tilde{\nu}}{2}\xi^2 \pm i\gamma\xi + O(\xi^3),$$

$$P_{\pm} = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} (1 + O(\xi)).$$

We define $S(t)$ and $S_{\pm}(t)$ by

$$S(t) = S_+(t) + S_-(t),$$

$$S_{\pm}(t) = \mathcal{F}^{-1} \hat{S}_{\pm}(t) \mathcal{F},$$

$$\hat{S}_{\pm}(t) = \frac{1}{2} e^{-\frac{\nu+\tilde{\nu}}{2}\xi^2 t \pm i\gamma\xi t} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

Clearly, $e^{-tL_0} P_0$ has the same estimate as that for $e^{-tL} P_1$ such as (22). Furthermore, $e^{-tL_0} P_0$ is approximated by $S(t)$ in the following way. We define Π_0 by

$$\Pi_0 U_0 = {}^\top(\langle \phi_0 \rangle, \langle M_0 \rangle) \quad \text{for } U_0 = {}^\top(\phi_0, M_0).$$

Note that $\Pi_0 P_0 = P_0 \Pi_0 = P_0$.

We denote by $U^{(0)}(t) = {}^\top(\phi^{(0)}(x_1, t), M^{(0),1}(x_1, t))$ the solution of the following integral equation:

$$U^{(0)}(t) = S(t) \Pi_0 U_0 + \int_0^t S(t-\tau) \partial_{x_1} G(U^{(0)}(\tau)) d\tau. \quad (30)$$

We see from (29) that $U(t)$ is written as

$$U(t) = e^{-tL_0} P_0 U_0 + \int_0^t e^{-(t-\tau)L_0} P_0 \partial_{x_1} (G(U) + \tilde{G})(\tau) d\tau. \quad (31)$$

we have the following estimates for $U^{(0)}(t)$.

Proposition 3.5 *If $\|U_0\|_{H^2 \cap L^1} \ll 1$, then (30) has a unique solution $U^{(0)}(t)$ that satisfies*

$$\|\partial_{x_1}^k U^{(0)}(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|U_0\|_{H^2 \cap L^1}, \quad k = 0, 1, 2, \quad (32)$$

$$\|\partial_{x_1}^k U^{(0)}(t)\|_{L^\infty} \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|U_0\|_{H^2 \cap L^1}, \quad k = 0, 1. \quad (33)$$

We have the following estimate for $U(t) - U^{(0)}(t)$.

Theorem 3.6 *If $\|U_0\|_{H^2 \cap L^1} \ll 1$, then*

$$\|\partial_{x_1}^k (U(t) - U^{(0)}(t))\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta} \|U_0\|_{H^2 \cap L^1}, \quad k = 0, 1,$$

for any $\delta > 0$.

Proof of Theorem 2.2. It suffices to show that $\|\partial_{x_1}^k (U^{(0)} - \chi_+ \mathbf{b}_+ - \chi_- \mathbf{b}_-)(t)\|_{L^2}$ for $k = 0, 1$, where $\mathbf{b}_\pm = {}^\top(1, \pm 1) \in \mathbb{R}^2$. Here $\chi_\pm = \chi_\pm(x_1, t)$ is the diffusion waves given in (10)-(12) with $c = \frac{1}{2}(a+b)$, $a = -\frac{\rho_* P''(\rho_*)}{2\gamma}$, $b = -\gamma$. We follow the arguments in [7, 6]. We write U_0 as

$$U_0 = U_{0+} + U_{0-},$$

where

$$U_{0\pm} = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} \Pi_0 U_0 = \frac{1}{2} \langle \phi_0 \pm \frac{1}{\rho_* \gamma} m_0^1 \rangle \mathbf{b}_\pm.$$

It then follows that

$$U^{(0)}(t) = S_+(t)U_{0+} + S_-(t)U_{0-} + I_{1,+}(t) + I_{1,-}(t),$$

where

$$I_{1,\pm}(t) = \int_0^t S_\pm(t-\tau) \partial_{x_1} \begin{pmatrix} 0 \\ a(\phi^{(0)})^2 + b(M^{(0),1})^2 \end{pmatrix} d\tau.$$

We write $I_{1,\pm}(t)$ as

$$I_{1,\pm} = \pm \frac{1}{2} \int_0^t e^{-(t-\tau)L_\pm} \partial_{x_1} (a(\phi^{(0)})^2 + b(M^{(0),1})^2) d\tau \mathbf{b}_\pm,$$

where

$$e^{-tL_\pm} u_0 = \mathcal{F}^{-1} [e^{(-\frac{\nu+\beta}{2}\xi^2 \pm i\gamma\xi)t} \hat{u}_0].$$

We note that e^{-tL_\pm} satisfies the same estimates as those for $S_\pm(t)$.

We define $V(t) = {}^\top(\eta(t), \zeta(t))$ by

$$\begin{aligned} U^{(0)}(t) &= \chi_+(t) \mathbf{b}_+ + \chi_-(t) \mathbf{b}_- + V(t) \\ &= \begin{pmatrix} \chi_+ + \chi_- + \eta \\ \chi_+ - \chi_- + \zeta \end{pmatrix}, \end{aligned}$$

and introduce

$$Y(t) = \sup_{0 \leq \tau \leq t} \{ (1+\tau)^{\frac{1}{2}} \|V(\tau)\|_{L^2} + (1+\tau) \|\partial_{x_1} V(\tau)\|_{L^2} \}.$$

We write

$$\begin{aligned}(\phi^{(0)})^2 &= \chi_+^2 + \chi_-^2 + 2\chi_+\chi_- + \sigma_1\eta, \\ (M^{(0),1})^2 &= \chi_+^2 + \chi_-^2 - 2\chi_+\chi_- + \sigma_2\zeta,\end{aligned}$$

where $\sigma_1 = \chi_+ + \chi_- + \phi^{(0)}$ and $\sigma_2 = \chi_+ - \chi_- + M^{(0),1}$. It then follows that $I_{1,\pm}(t)$ is written in the following forms

$$\begin{aligned}I_{1,\pm}(t) &= \pm \frac{1}{2} \int_0^t e^{-(t-\tau)L_{\pm}} \partial_{x_1} \left((a+b)(\chi_+^2 + \chi_-^2) + 2(a-b)\chi_+\chi_- \right. \\ &\quad \left. + a\sigma_1\eta + b\sigma_2\zeta \right) d\tau \mathbf{b}_{\pm}.\end{aligned}$$

Since χ_{\pm} satisfies

$$\chi_{\pm}(t) = e^{-tL_{\pm}} \chi_{0\pm} \pm \frac{a+b}{2} \int_0^t e^{-(t-\tau)L_{\pm}} \partial_{x_1}(\chi_{\pm}^2)(\tau) d\tau,$$

where $\chi_{0\pm} = \chi_{\pm}(0)$, we see that

$$\begin{aligned}V(t) &= U^{(0)}(t) - \chi_+(t)\mathbf{b}_+ - \chi_-(t)\mathbf{b}_- \\ &= S_+(t)(U_{0+} - \chi_{0+}\mathbf{b}_+) + S_-(t)(U_{0-} - \chi_{0-}\mathbf{b}_-) + I_{1,+} + I_{1,-} \\ &\quad - \frac{a+b}{2} \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\chi_+^2)(\tau) d\tau \mathbf{b}_+ + \frac{a+b}{2} \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\chi_-^2)(\tau) d\tau \mathbf{b}_- \\ &= S_+(t)(U_{0+} - \chi_{0+}\mathbf{b}_+) + S_-(t)(U_{0-} - \chi_{0-}\mathbf{b}_-) \\ &\quad + \frac{1}{2}(a+b) \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\chi_-^2)(\tau) d\tau \mathbf{b}_+ \\ &\quad - \frac{1}{2}(a+b) \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\chi_+^2)(\tau) d\tau \mathbf{b}_- \\ &\quad + (a-b) \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\chi_+\chi_-)(\tau) d\tau \mathbf{b}_+ \\ &\quad - (a-b) \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\chi_+\chi_-)(\tau) d\tau \mathbf{b}_- \\ &\quad + \frac{1}{2}a \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\sigma_1\eta)(\tau) d\tau \mathbf{b}_+ \\ &\quad - \frac{1}{2}a \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\sigma_1\eta)(\tau) d\tau \mathbf{b}_- \\ &\quad + \frac{1}{2}b \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\sigma_2\zeta)(\tau) d\tau \mathbf{b}_+ \\ &\quad - \frac{1}{2}b \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\sigma_2\zeta)(\tau) d\tau \mathbf{b}_-.\end{aligned}$$

It then follows that

$$\|\partial_{x_1}^k V(t)\|_{L^2} \leq \sum_{j=\pm} \|\partial_{x_1}^k S_j(t)(U_{0j} - \chi_{0j}\mathbf{b}_j)\|_{L^2}$$

$$\begin{aligned}
& + C_1 (\|\partial_{x_1}^k w_+(t)\|_{L^2} + \|\partial_{x_1}^k w_-(t)\|_{L^2}) \\
& + C_2 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+} \partial_{x_1}(\chi_+ \chi_-)(\tau)\|_{L^2} d\tau \\
& + C_3 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-} \partial_{x_1}(\chi_+ \chi_-)(\tau)\|_{L^2} d\tau \\
& + C_4 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+} \partial_{x_1}(\sigma_1 \eta)(\tau)\|_{L^2} d\tau \\
& + C_5 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-} \partial_{x_1}(\sigma_1 \eta)(\tau)\|_{L^2} d\tau \\
& + C_6 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+} \partial_{x_1}(\sigma_2 \zeta)(\tau)\|_{L^2} d\tau \\
& + C_7 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-} \partial_{x_1}(\sigma_2 \zeta)(\tau)\|_{L^2} d\tau \\
& =: \sum_{j=\pm} \|\partial_{x_1}^k S_j(t)(U_{0j} - \chi_{0j} \mathbf{b}_j)\|_{L^2} + \sum_{j=1}^7 I_j.
\end{aligned}$$

where

$$\begin{aligned}
w_{\pm}(t) &= \int_0^t e^{-(t-\tau)L_{\pm}} \partial_{x_1}(\chi_{\mp}^2)(\tau) d\tau, \\
C_1 &= \frac{1}{2}|a+b|, \quad C_2 = C_3 = |a-b|, \quad C_4 = C_5 = \frac{1}{2}|a|, \quad C_6 = C_7 = \frac{1}{2}|b|.
\end{aligned}$$

Since

$$\begin{aligned}
& \int_{\mathbb{R}} (U_{0\pm} - \chi_{0\pm} \mathbf{b}_{\pm}) dx_1 \\
&= \left[\frac{1}{2} \int_{\Omega} \left(\phi^{(0)} \pm \frac{1}{\rho_* \gamma} m_0^1 \right) dx - \int_{\mathbb{R}} \chi_{0\pm} dx_1 \right] \mathbf{b}_{\pm} = 0,
\end{aligned}$$

we have

$$\|\partial_{x_1}^k S_{\pm}(t)(U_{0\pm} - \chi_{0\pm} \mathbf{b}_{\pm})\|_{L^2} \leq C t^{-\frac{1}{2} - \frac{k}{2}} \|u_0\|_{L_{1/2}^1}.$$

As for I_1 , we apply the estimates for w_{\pm} by T.-P. Liu [8] (see also [6, Lemma 4.2]) to obtain

$$I_1 \leq C(1+t)^{-\frac{1}{2} - \frac{k}{2}} \|u_0\|_{H^2 \cap L^1}^2.$$

We next estimate I_2 . For $1 \leq p \leq \infty$ and $l \geq 0$, we have

$$\|\partial_x^l (\chi_+ \chi_-)(t)\|_{L^1} \leq C e^{-ct} \|u_0\|_{H^2 \cap L^1}^2. \quad (34)$$

It then follows from (34) that

$$I_2 \leq C(1+t)^{-\frac{3}{4} - \frac{k}{2}} \|u_0\|_{H^2 \cap L^1}^2.$$

Similarly, we have $I_3 \leq C(1+t)^{-\frac{3}{4} - \frac{k}{2}} \|u_0\|_{H^2 \cap L^1}^2$.

As for I_4 , we have

$$\begin{aligned}
I_4 &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} \|\sigma_1 \eta(\tau)\|_{L^1} d\tau \\
&\quad + C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}} \|\partial_{x_1}^k (\sigma_1 \eta)(\tau)\|_{L^2} d\tau \\
&\quad + C \int_0^t e^{-c_0(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_{x_1}^k (\sigma_1 \eta)(\tau)\|_{L^2} d\tau \\
&=: I_{41} + I_{42} + I_{43}.
\end{aligned}$$

By applying Proposition 3.5 and the following estimate

$$\|\partial_{x_1}^k \chi_{\pm}(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|u_0\|_{L^1}, \quad (35)$$

we see that $\|\sigma_1(\tau)\|_{L^2} \leq C(1+\tau)^{-\frac{1}{4}} \|u_0\|_{H^2 \cap L^1}$. Since $\|\sigma_1 \eta\|_{L^1} \leq \|\sigma_1\|_{L^2} \|\eta\|_{L^2}$, we have

$$\begin{aligned}
I_{41} &\leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} Y(t), \\
I_{42} &\leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} Y(t), \\
I_{43} &\leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} Y(t).
\end{aligned}$$

We thus obtain $I_4 \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} Y(t)$. We can obtain the estimates for I_5, I_6, I_7 in a similar manner. It then follows that if $\|u_0\|_{H^2 \cap L^1} \ll 1$, we have

$$\|\partial_x^k V(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} \quad (36)$$

for $k = 0, 1$.

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